(7 pages)

Reg. No. : ....

Code No. : 5853 **Sub. Code : PMAM 42**

M.Sc. (CBCS) DEGREE EXAMINATION,
APRIL 2020.

Fourth Semester

Mathematics — Core

COMPLEX ANALYSIS

(For those who joined in July 2017 onwards)

Time : Three hours Maximum : 75 marks

PART A — ($10 \times 1 = 10$ marks)

Answer ALL questions.

Choose the correct answer :

1. If $f(z)$ is a real function of a complex variable then

   (a) the derivative does not exist

   (b) the derivative exists

   (c) either the derivative is zero or the derivative does not exist

   (d) the derivative is zero

2. The formula $\frac{1}{R} = \lim_{n\to\infty} \sup \sqrt[n]{|a_n|}$ is known as

(a) Hadamard's formula

(b) Cauchy's formula

(c) Abel's formula

(d) Rouche's formula

3. If $\int_{\gamma} f(z)\, dz = 2+i$ then $\int_{-\gamma} f(z)\, dz$ is

(a) $2 - i$ (b) $2 + i$

(c) $-2 - i$ (d) 0

4. If $w = S(z) = \frac{az+b}{cz+d}$ then $S^{-1}(w)$ is

(a) $\frac{dw-b}{-cw+a}$ (b) $\frac{dw-b}{cw-a}$

(c) $\frac{dw-b}{-cw-a}$ (d) z^{-1}

5. $\int_{|z|=1} \frac{e^z}{z}\, dz$ is

(a) 0 (b) 1

(c) 2π (d) ∞

6. The index of the point a w.r.t. the curve γ is

(a) $\frac{1}{2\pi i}\int_{\gamma}(z-a)\,dz$ (b) $\frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z-a}$

(c) $\frac{1}{2\pi i}\int_{\gamma}dz$ (d) $\int_{\gamma}\frac{dz}{z-a}$

7. "If $f(z)$ is a polynomial of degree > 0 then $f(z)=0$ must have a root" — This result is known as

(a) Liouville's theorem

(b) Morera's theorem

(c) Fundamental theorem of algebra

(d) Cauchy's theorem

8. If $f(z)$ is analytic and non constraint is a region Ω then

(a) $f(z)$ has no maximum in Ω

(b) $f(z)$ has maximum in Ω

(c) $|f(z)|$ has maximum in Ω

(d) $|f(z)|$ has no maximum in Ω

9. The residue of the function $\dfrac{ez}{(z-a)(z-b)}$ at the pole a is

(a) $\dfrac{e^a}{b-a}$ (b) $\dfrac{e^a}{a-b}$

(c) $\dfrac{e^b}{a-b}$ (d) $\dfrac{e^b}{b-a}$

10. Residue of $\dfrac{z+1}{z^2-2z}$ at $z=0$ is

(a) $\dfrac{1}{2}$ (b) 0

(c) 3/2 (d) – 1/2

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) State and prove Lucas's theorem.

Or

(b) Show that an analytic function cannot have a constant absolute value without reducing to a constant.

12. (a) Find the linear transformation which carries $0, i, -i$ into $1, -1, 0$.

Or

(b) Show that the transformation $z \to \bar{z}$ is not a linear transformation.

13. (a) If the piecewise differentiable closed curve γ does not pass through the point a, prove that the value of the integral $\int_\gamma \frac{dz}{z-a}$ is a multiple of $2\pi i$.

Or

(b) As a function of a, prove that the index $n(r, a)$ is constant in each of the regions determined by γ and zero in the unbounded region.

14. (a) State and prove Morera's theorem.

Or

(b) State and prove the fundamental theorem of algebra.

15. (a) State and prove Rouche's theorem.

Or

(b) State and prove the residue theorem.

PART C — ($5 \times 8 = 40$ marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) Derive the Cauchy-Riemann differential equations which must be satisfied by the real and imaginary part of any analytic function and hence show that $|f'(z)|^2$ is the Jacobian of u and v w.r.t. x and y where $f(z) = u(z) + iv(z)$.

Or

(b) State and prove Abel's limit theorem.

17. (a) Obtain a necessary and sufficient condition under which a line integral depends only on the end points.

Or

(b) Compute $\int\limits_{|z|=1} |z-1| . |dz|$.

18. (a) If the function $f(z)$ is analytic in a rectangle R, prove that $\int\limits_{\partial R} f(z)\, dz = 0$.

Or

(b) Let $f(z)$ be analytic on the set R' obtained from a rectangle R by omitting a finite number of interior points ζ_j. If $\lim\limits_{z \to \zeta_i} (z - \zeta_i)\ f\ (z) = 0$ for all i, prove that $\int\limits_{\partial R} f(z)\, dz = 0$.

19. (a) Suppose that $\phi(\zeta)$ is continuous on the arc γ. Prove that $Fn(z) = \int_{\gamma} \frac{\phi \zeta \, d\zeta}{(\zeta - z)\, n}$ is analytic in each of the regions determined by γ and derivatives is $Fn'(z) = n\, Fn + 1(z)$.

Or

(b) (i) State and prove the maximum principle.

(ii) State and prove the lemma of Schwarz.

20. (a) State and prove the argument principle.

Or

(b) Evaluate $\int_0^{\infty} \frac{x \sin x}{x^2 + a^2}\, dx$, a real.